For simplicity we assume that the system is represented by a regular cubic bed of spheres of equal diameter (Fig. 1). The strains of this regular system of spheres have such

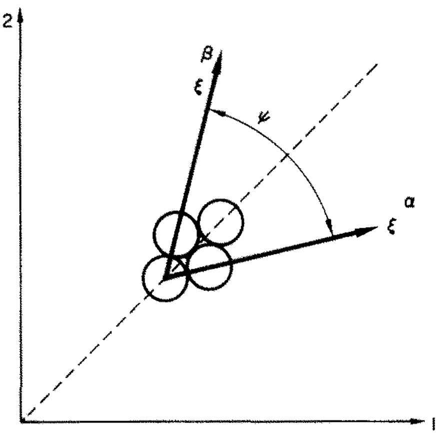

FIG. 1.

features, that at every moment of time the relative displacements of the centers of these spheres are perpendicular to the lines connecting the centers. Naturally, the position of the lines connecting these centers is changed during the deformation of the bed of spheres.

In case the continuum description is introduced, it will be necessary to characterize the macropoint with the directors, coinciding with the lines of sphere centers, that is one has to analyze the continuum type, considered earlier by Ericksen and Truesdell [1].

In the plane case there are two directors ξ_i^α and ξ_i^β, the angle between them being equal to ψ. If the displacement velocity is u_i, then the above kinematical condition of the absence of relative displacements along the directors will have a form

$$\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)\xi_j^\alpha\xi_i^\alpha = 0, \qquad \sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)\xi_j^\beta\xi_i^\beta = 0. \tag{2.1}$$

Decomposing the tensor of displacement gradient $\partial u_i/\partial x_j$ into antisymmetrical and symmetrical ε_{ij} parts, we transform the conditions (2.1) to the following

$$\sum_{i,j}\varepsilon_{ij}\xi_i^\alpha\xi_j^\alpha = 0, \qquad \sum_{i,j}\varepsilon_{ij}\xi_i^\beta\xi_j^\beta = 0. \tag{2.2}$$

Further, the conditions (2.2) are equivalent to

$$\varepsilon = \Lambda\varepsilon_{12}, \qquad \Lambda = -2\cos\psi. \tag{2.3}$$

We use here such a coordinate system 1, 2 that the bisectrix of coordinate angle coincides with the bisectrix of the angle ψ (see Fig. 1).

According to the equation (2.3) the volume strain rate $\varepsilon = \varepsilon_{11}+\varepsilon_{22}$ is connected with the shear rate ε_{12}, that is we encounter the phenomenon of dilatancy, described by Reynolds [2] in 1895. One may interpret the quantity Λ as the rate of dilatancy. If $\psi = \pi/2$, shear failure along the directors without volume changes is possible. If $\psi = \pi/3$ the compression (and associated shear) is prohibited—the medium becomes a rigid one.

The changes of the angle ψ, associated with the relative position of the directors ξ_i^α, ξ_i^β, are defined by the condition $\mathrm{d}\xi_i^\alpha = \{u_i(x_i+\xi_i^\alpha, t)-u_i(x_i, t)\}\,\mathrm{d}t$, that may be transformed to the following

$$\frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}t} = 2\varepsilon_{12}\sin\psi. \tag{2.4}$$

We underline the fact that dilatancy is a kinematical restriction.

In considered continuum there is a restriction for stresses. On the plane cross-sections with the directors as a normal, for instance ξ_i^α, the shear stress τ^α and the normal stress σ^α are related to each other by Coulomb rule $\tau^\alpha\theta = k\sigma^\alpha$, where $\theta = \operatorname{sgn}\tau^\alpha$, k is the friction coefficient. This condition has a form

$$\sigma_{12}(\operatorname{sgn}\sigma_{12})\sin\psi = k(\sigma+\sigma_{12}\cos\psi) \qquad (\theta k \operatorname{ctg}\psi \neq 1) \tag{2.5}$$

where $\sigma = \sigma_{11} = \sigma_{22}$ according to the symmetry in elated coordinate system. The limitation (2.5) in a general case (when $\psi \neq \pi/2$) is distinct from the usual Coulomb law for continuous media.

3. ELASTIC MEDIA WITH LOCAL SOLID FRICTION

If particles (grains) have different sizes and form an irregular bed, then one can imagine such a situation that the strains of the materials are determined by the small elasticity of grains, and there is slippage with solid friction among some particles. One can try to describe such a case with the help of a continuum, at every macropoint of which there are at least two types of motions, and to interpret the relative motion as a "slip".

The total stresses $p_{kl} = \sigma_{kl}+\tau_{kl}$ are decomposed into two parts: σ_{kl} and τ_{kl}. The elastic stresses σ_{kl} are connected with mean strains e_{ij} according to the elastic law (with modulus K and G). The plastic stresses τ_{kl} also cause elastic strains e_{ij}^*, unequal, however, to mean strains. In the last case the elastic moduli are K^*, G^* ($K, G \gg K^*, G^*$). The differences $\varepsilon_{ij} = e_{ij}-e_{ij}^*$ form the plastic strains. The rates of plastic strains are unequal to zero and connected with stresses through the following isotropic rule

$$\dot{\varepsilon}_{ij} = [-\tfrac{2}{3}(1+\Lambda\alpha)\delta_{ij}\delta_{kl}+2\delta_{ik}\delta_{j}e]\dot{\lambda}\tau_{kl} \qquad (K = 1, 2) \tag{3.1}$$

if the limit condition holds

$$\sqrt{(J_2')}+\alpha J_1 = 0. \tag{3.2}$$

Here J_1 is the first, J_2' the second (deviatoric) invariant of the stress tensor. The function λ, defined as positive, is considered as an additional unknown quantity.

From the constitutive relations (3.1) and the limit condition it follows that volume and shear strains take place simultaneously:

$$I_1 = \Lambda\sqrt{(I_2')} \tag{3.3}$$

where I_1 is the first invariant of the strain rate tensor, I_2' the second invariant of the strain rate deviator tensor. Therefore quantity Λ may be interpreted as dilatancy rate.

The mathematical model is completed by the equation of linear momentum. In the case of small perturbations it has the following form

$$\rho\frac{\partial u_i}{\partial t}+\frac{\partial p_{ij}}{\partial x_j}=0, \qquad \dot{e}_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{3.4}$$

Here u_i is the mean velocity of particle displacements. Let us compare the suggested model and the elastic material with microstructure, considered by Mindlin [3]. It is seen, that excluding asymmetrical features of the Mindlin model, the difference is the following one. According to Mindlin micromotions are determined by the equation of linear momentum; that is the special motion of micromaterial is connected with its inertial properties. The model considered here is associated to the case of local static equilibrium, when the inertial forces are negligible compared with solid friction forces acting on the micromaterial.

The set of equations (3.1)–(3.4) was used for approximate analysis of the seismic wave propagation in soils and rocks [4], the following consideration being realized. The usual limit condition (3.2) is valid only in case of the compressive normal stress: $\operatorname{sgn} J_1 < 0$. In case of periodic motion the sign of normal stresses may be changed. However in granular media during the dilate strain there are local slips with solid friction, although generally speaking, on other contacts between grains then during compression. For the purpose of checking this effect we shall determine the coefficient α in the form: $\alpha = -\alpha_0 \operatorname{sgn} J_1$, where α_0 has only positive defined values.

One may note that just the dilatant relation (3.3) introduces the variable normal plastic stresses in shear waves. Approximate evaluation on the basis of the simple method of harmonic linearization, which was used generally for dynamical analysis of systems with solid friction, shows the following results [4]. If the role of solid friction is small then the wave of fixed frequency propagates without dispersion (as in usual elastic media), and the attenuation coefficient associated with transmitted distance, is proportional to the frequency. Such a behavior of waves corresponds to the familiar features of seismic waves in dry soils and rocks.

4. ASYMMETRICAL EFFECTS IN TURBULENT FLOWS

For the averaged description of turbulent flows let us assume the usual continuum demand of possible choosing of such an elementary volume ΔV, whose dimensions are much larger than the internal scale of microstructure (represented by the turbulent eddies), but much smaller than the characteristic lengths of the flow considered. We assume that microflows are described by the Navier–Stokes equations. Then for the establishment of Reynolds macroequations it is necessary to integrate the balance equations of mass, linear momentum and moment of momentum over the volume. We demand that the balance equation for the moment of momentum for microflows has to be the consequence of the balance equation of linear momentum (the tensors, included in Navier–Stokes equations, are symmetric). That is the moment of momentum equation has to coincide with the "vorticity equation" for a viscous fluid. From this requirement it follows the definition of the inertial moment of fluid differential volume and the possibility to interpret the diffusive transfer of vorticity as a couple-stress in usual viscous fluid.

The procedure of integration leads to solely choosing of the laws of averaging for the quantities included in microequations. Thus, averaged density and averaged linear momentum are introduced as averaged quantities over the volume. The mean velocity V_i is equal to the averaged linear momentum divided by the averaged density and therefore also is the quantity averaged over the volume. In the same way we introduce the averaged angular velocity of turbulent eddies, distinct in general case from angular velocity of mean translation motion. In nonisotropic turbulence the existence of statistical orientation of turbulent eddies introduces the additional averaged moment of momentum.

The averaging procedure of flows of linear momentum and of moment of momentum, stresses, denoted by the symbol $\langle \ldots \rangle_j$, is fulfilled over the plane cross-sections of elementary macrovolume. For nonisotropic turbulence the results of such a procedure depend on the orientation of the plane compared with the characteristic vector of anisotropic turbulent structure. Therefore, generally speaking, Reynolds stress tensor z_{ij}, that is the averaged pulsating transfer of linear momentum, may be asymmetric. For nonisotropic turbulence the couple-stresses μ_{ij}, connected with pulsating transfer of micromoment of momentum equal to $I_{ik}\Phi_k$, are also essential. Here I_{ik} is the moment of inertia and Φ_k is the angular velocity of microparticles.

The set of averaged equations for flows of turbulent incompressible fluid has the following form:

$$\frac{\partial V_i}{\partial x_j} = 0 \tag{4.1}$$

$$\rho\frac{\partial V_i}{\partial t} + \rho\frac{\partial}{\partial x_j}V_iV_j = \frac{\partial T_{ij}}{\partial x_j} + \frac{\partial z_{ij}}{\partial x_j} \tag{4.2}$$

$$\frac{\partial[\mathbf{I}^0 . (\vec{\Omega}+\vec{\omega})]}{\partial t} + V_j\frac{\partial}{\partial x_j}[\mathbf{I}^0 . (\vec{\Omega}+\vec{\omega})]_i = z_{[ij]} + \frac{\partial \mu_{ij}}{\partial x_j}. \tag{4.3}$$

Here $\vec{\Omega} = \frac{1}{2}\operatorname{rot}\vec{V}$, T_{ij} are averaged viscous stresses, $z_{ij} = -\langle \rho v_i v_j \rangle$ — Reynolds stresses, $z_{[ij]}$ their antisymmetric part, $\mu_{ij} = -\langle I_{ik}\Phi_k v_i \rangle_j$, v_j — pulsating velocity, I^0_{ik}—moment of inertia, ω—effective angular velocity of pulsating rotations of the microparticles. The equation for the moment of inertia $\mathbf{I}^0$ of the considered fluid element can be written as

$$(\vec{\Omega}+\vec{\omega}) . \left(\frac{\partial \mathbf{I}^0}{\partial t} + V_j\frac{\partial \mathbf{I}^0}{\partial x_j}\right) + \mathbf{I}^0 . (\Omega_j + \omega_j)\frac{\partial \vec{V}}{\partial x_j} = 0 \tag{4.4}$$

As is usual in semiempirical theories, the set of equations (4.1)–(4.4) is supplemented by the constitutive relations

$$\begin{aligned} z_{ij} - z_{[ij]} &= A_{ijkl}\left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k}\right), \qquad \mu_{ij} = B_{ijkl}\frac{\partial(\Omega_k + \omega_k)}{\partial x} \\ T_{ij} &= -p\delta_{ij} + \nu\rho\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right), \qquad z_{[ij]} = D_{ijkl}E_{klm}\omega_m \end{aligned} \tag{4.5}$$

Here p is the pressure, ν the kinematic viscosity, E_{klm} the Levi-Civita axial tensor. The transfer coefficients A_{ijkl}, B_{ijkl}, D_{ijkl}, depend on the microstructure of the turbulent fluid. For their evaluation it is necessary to introduce the hypotheses of mixing kinetics in turbulent flow. It is useful to note that the Prandtl's idea [6] of impulse transfer and Taylor's idea [7] of vorticity transfer are complementary to each other in the case of account of asymmetrical effects.

In this text we omit the possibility of asymmetric averaged viscous tensor T_{ij} and of the effects of nonhomogeneous fields of pulsating impulse and viscous microstresses [14].

5. ASYMMETRIC HYDRODYNAMICS OF DILUTE SUSPENSIONS

The movement of a fluid particle can be decomposed into three elementary parts: translation, rotation and deformation. The suspended solid particle (of spherical form, for simplicity) has, however, its own translational and angular velocities (that is connected with distinct inertia of the suspended particle) and besides perturbs the deformation process of the surrounding fluid. The third effect was considered by Einstein [8]. It is accounted by the introduction of effective viscosity $\mu_e = \mu(1+2{\cdot}5(1-m))$, see also [15].

To accounting for the difference between translating and rotationary motions of solid and fluid particles it is necessary to consider the balance equations of mass, linear momentum and moment of momentum separately for solid and fluid phases.

The mass balance equations for the phases have the forms:

$$\frac{\partial m\rho_1}{\partial t}+\frac{\partial m\rho_1 v_j^{(1)}}{\partial x_j} = 0, \qquad \frac{\partial(1-m)\rho_2}{\partial t}+\frac{\partial(1-m)\rho_2 v_j^{(2)}}{\partial x_j} = 0 \tag{5.1}$$

Here m is the saturation of elementary volume with the fluid; superscript (1) denotes the fluid phase and superscript (2) the solid one.

The linear momentum equations for phases can be written as

$$m\rho_1\frac{\mathrm{d}_1 v_i^{(1)}}{\mathrm{d}t} = -m\frac{\partial p}{\partial x_i}+\frac{\partial m(\tau_{ji}+\sigma_{ji}^a)}{\partial x_j}-mR_i \tag{5.2}$$

$$(1-m)\rho_2\frac{\mathrm{d}_2 v_i^{(2)}}{\mathrm{d}t} = -(1-m)\frac{\partial p}{\partial x_i}+mR_i \tag{5.3}$$

Here p is a mean pressure in the fluid, τ_{ij} is the symmetrical part of viscous stresses, σ_{ij}^a the antisymmetrical one. We underline that the introduction of this asymmetrical part is connected with perturbations in stress-field in viscous fluid due to own rotations of suspended particles. The force of interaction between the phases $R_i = a(v_i^{(1)}-v_i^{(2)})-bE_{kij}(v_j^{(1)}-v_j^{(2)}) \times(\omega_k-\Omega_k^{(1)})$ can be evaluated by assuming the microstationary character of interaction of suspended particles with surrounding fluid. The first term is associated with Stokes resistance, the second one with the lift force. Here $\vec{\Omega}^{(1)} = \frac{1}{2}\operatorname{rot}\vec{v}^{(1)}$ is the mean angular velocity of fluid element, $\vec{\omega}$ is the angular velocity of own rotation of the sphere.

From the equation for moment of momentum of the suspension and the analogous equation for moment of momentum of translational motion it is possible to find the field equation for the internal moment of momentum which is reduced to the following one,

$$(1-m)\rho_2\frac{\mathrm{d}_2 L_k}{\mathrm{d}t} = mE_{ijk}\sigma_{ij}^a. \tag{5.4}$$

Here $(1-m)\rho_2 L_k = N\rho_2 J\omega_k$, $\rho_2 J = (\frac{8}{15})\pi r^5\rho_2$ is the moment of inertia of sphere with radius r, $N = (1-m)(\frac{4}{3}\pi r^3)^{-1}$ is the number of suspended particles per unit volume. Further, we shall consider only plane flows.

The equation for internal moment of momentum of solid particles has the form:

$$\rho_2 J\frac{\mathrm{d}_2\omega_3}{\mathrm{d}t} = -\gamma(\omega_3-\Omega_3), \qquad \gamma = 8\pi r^3\mu \tag{5.5}$$

In the considered case the constitutive relation for the antisymmetrical part of the viscous stress tensor reduces to

$$mE_{3ij}\sigma^{a}_{ij} = -N\gamma(\omega_3 - \Omega^{(1)}_3) \tag{5.6}$$

as it follows from equations (5.4) and (5.5). In a more general approach as is usual in asymmetrical hydrodynamics [9–11] it is necessary to introduce the couple-stresses. By the way in the paper [12] the couple-stresses were related to diffusion transfer of rotationary particles through the fluid. However, we prefer to describe the relative translation of phases by the separate equations of linear momentum.

If we linearize the set of equations (5.1)–(5.5) and introduce the potentials of motion [13], then it will be possible to write the equation for shear waves

$$\begin{aligned} &\tau_\omega \tau_a \frac{\partial^2}{\partial t^2}\left(\frac{\partial \psi}{\partial t} - \frac{\mu_e+\delta}{m_0\rho_1^0}\nabla^2\psi\right) + \tau_\omega \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t} - \frac{\mu_e+\delta}{\rho_0}\nabla^2\psi\right) \\ &+ \tau_a \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t} - \frac{\mu_e}{m_0\rho_1^0}\nabla^2\psi\right) + \left(\frac{\partial \psi}{\partial t} - \frac{\mu_e}{\rho_0}\nabla^2\psi\right) = 0. \end{aligned} \tag{5.7}$$

Here ψ is the potential of shear motion, $\tau_a = (\frac{2}{9})(r^2/\mu)(\rho_1^0\rho_2^0/\rho_0)$ the time of translational relaxation, $\tau_\omega = (\frac{1}{15})(r^2/\mu)\rho_2^0$ the time of angular relaxation, $\rho_0 = m_0\rho_1^0 + (1-m_0)\rho_2^0$ is the effective mean density for equilibrium flows. The analysis of the operators in the equation (5.7) leads to an evaluation of the flows with different characteristic times. Essentially, in "frozen" flow (the limit deviation from Einstein's case due to inertial forces) the suspension behaves as a viscous fluid, but with the effective density equal to the mass of fluid phases per unit volume $m_0\rho_1^0$ and with effective viscosity equal to $\mu_e + \delta$, where $\delta = \frac{3}{2}\mu(1-m)$.

The effect of rotation is essential for the suspension of solid particles in gas media. The relaxation times τ_a and τ_ω has of the same order $O(r^2)$ when the radius of the particles r approaches zero.

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(*Received* 9 *September* 1968)

Абстракт—Приведены четыре примера сплошных сред, для описания поведения которых необходимо прибегать к анализу их микроструктуры. Рассмотрены особенности деформаций и напряжений ориентированного континуума, моделирующего поведение регулярной упаковски жестких сфер, на контактах между которыми действует сухое трение. Характерный для такой среды кинематический эффект дилатансии вводится далее в модель упругой среды с локальными проявлениями сухого трения. Рассмотрение перехода от микроуравнений Навье-Стокса к макроуравнениям Рейнольдса в случае асимметричного турбулентного потока жидкости показывает, что тензор рейнольдстовых напряжений, вообще говоря, несимметричен, и должно быть выведено уравнение баланса момента количества движения. Модель двух взаимопроникающих континуумов (жидкой и твердой фаз) для разбавленных суспензий твердых частиц обобщает анализ Эйнштейна для ламинарных безинерционных потоков на случай инерционной релаксации из-за различия поступательных и вращате льных движений фаз.